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On the Problem of Existence of Holomorphic First Integrals

MASSIMO VILLARINI

Dipartimento di Matematica "U. Dini," Università degli Studi di Firenze, V.le Morgagni 67/A, 50134 Firenze, Italy

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1. INTRODUCTION

Let us consider a differential equation

$$\begin{aligned} \dot{x} &= B(x,y) \\ \dot{y} &= -A(x,y), \end{aligned} \tag{1.1}$$

where $A(x, y)$, $B(x, y)$ are real analytic functions in a neighbourhood of $O \in \mathbf{R}^2$ and $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, $t \in \mathbf{R}$.

The *complexification* of (1.1) is the holomorphic differential equation defined in a neighbourhood of $O \in \mathbf{C}^2$ by (1.1), when now $x, y \in \mathbf{C}$ and $\dot{x} = dx/dz$, $\dot{y} = dy/dz$, $z \in \mathbf{C}$. We refer to the *real dynamics* and to the *complex dynamics* of (1.1) as the topological configurations of the solutions of (1.1) respectively as a real and a complex differential equation. If $\mathscr{C}_m$ is an integral curve of the real dynamics passing through $m \in \mathbf{R}^2$, and $\mathscr{L}_m$ is the leaf of the foliation in $\mathbf{C}^2$ defined by the complex dynamics, $m \in \mathscr{L}_m$, then

$$\mathscr{C}_m \subset \mathscr{L}_m \cap \mathbf{R}^2,$$

where $\mathbf{R}^2$ is the submanifold obtained via the map $(x, y) \mapsto (x + i0, y + i0)$.

This paper concerns a classical problem: When does (1.1) admit a real analytic first integral? Of course to disregard trivial situations, *we will always suppose that $O \in \mathbf{R}^2$ is a singular point of* (1.1), hence $O \in \mathbf{C}^2$ is always a singular point of the complexification of (1.1), too.

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Let

$$\omega(x,y) = A(x,y)\,dx + B(x,y)\,dy \tag{1.2}$$

be the associated 1-form to (1.1): it is real analytic or holomorphic accordingly with the given interpretation of (1.1).

A *real analytic* (resp. *holomorphic*) *local first integral* of (1.1) is a (non-constant) analytic function

$$f\colon U \mapsto \mathbf{R}, \quad U \text{ neighbourhood of } O \in \mathbf{R}^2$$
$$(\text{resp. } f\colon U \mapsto \mathbf{C}, \quad U \text{ neighbourhood of } O \in \mathbf{C}^2)$$

such that

$$df \wedge \omega = 0.$$

Remark 1.1. The existence of a real analytic first integral of (1.1) is equivalent to the existence of a holomorphic first integral of the complexification of (1.1).

In fact, if $f(x, y)$ is a real analytic first integral of (1.1), locally we have $f(x, y) = \sum f_{ij} x^i y^j$: just considering $x, y \in \mathbf{C}$ in the Taylor expansion of $f(x, y)$ we obtain a holomorpic first integral of the complexification of (1.1). On the other hand, it is straightforward to check that if $g(x, y)$ is a holomorphic first integral of the complexification of (1.1), then the real analytic function

$$(\operatorname{Re} x, \operatorname{Re} y) \mapsto \operatorname{Re} g$$

is a first integral of (1.1). Hence throughout this paper we will always refer to the problem of existence of a real analytic first integral of (1.1) as the problem of existence of a holomorphic first integral of (the complexification of) (1.1).

A classical result is (we recall that $O \in \mathbf{R}^2$ is a *center* for the real dynamics of (1.1) if it is a singular point and it has a neighbourhood V such that $\phi V - \{0\}$ is filled by closed non-trivial integral curves; moreover with $J^1\omega$ we denote the first order jet of ω).

Center Theorem (Poincaré [1], Lyapunov [2]). *Let $J^1\omega$ be analytically conjugated to $x\,dx + y\,dy$. Then* (1.1) *has a local holomorphic first integral if and only if $O \in \mathbf{R}^2$ is a center for* (1.1).

Remark 1.2. It turns out that the topological hypothesis that $O \in \mathbf{R}^2$ is a center for the real dynamics of (1.1) is in itself sufficient for the existence of an infinitely smooth first integral (see [3]). Nevertheless, the same topological hypothesis is not sufficient for the existence of a holomorphic first integral. See the counterexample by Moussu in [4] and the remark at the end of Section 2.

The main result of this paper is that, roughly speaking, it is not possible to translate the only one known criterion for the existence of a holomorphic first integral into an algebraic algorithm. To give a statement, we need precise definitions. First we recall the definition of algebraic solvability with respect to a given algorithm.

Let W be a class of germs at $O \in \mathbf{R}^2$ of real analytic 1-forms. For any positive integer N, let W^N be the projection of W on the N-jet space, $\phi J^N(2)$, of 1-forms at $O \in \mathbf{R}^2$. Let us define the splitting of each W^N in the pairwise disjoint sets

$$W^N = A_+^N \cup A_-^N \cup A^N \tag{1.3}$$

$$\begin{cases} A_+^N := \{\hat\omega \in W^N \mid \omega \in W, J^N\omega = \hat\omega \Rightarrow \omega \text{ has a} \\ \qquad \text{holomorphic first integral}\} \\ A_-^N := \{\hat\omega \in W^N \mid \omega \in W, J^N\omega = \hat\omega \Rightarrow \omega \text{ does not have} \\ \qquad \text{a holomorphic first integral}\} \\ A^N = W^N - A_+^N \cup A^N. \end{cases} \tag{1.4}$$

A_+^N is the *positive set*, A_-^N is the *negative set*, A^N is the *neutral set*: it is the set of N-jets $\hat\omega$ for which we cannot decide if a 1-form ω such that only $J^N\omega = \hat\omega$ is known has or does not have a holomorphic first integral.

Definition 1.1. The problem of the existence of a holomorphic first integral is algebraically solvable in W if for every positive integer N

(1) A_+^N, A_-^N, A^N are the intersections of W^N with semialgebraic sets in $J^N(2)$

(2) the $\mathrm{cod}_{J^N(2)} A^N$ tends to infinity when $N \to \infty$.

We will not be able to solve the problem of the algebraic solvability as posed by Definition 1.1. The main difficulty is the characterization of the manifolds A_+^N, A_-^N, A^N. Instead we will consider and solve, in the negative sense, the less general problem of algebraic solvability with respect to the algorithm derived from the Topological Criterion by Mattei and Moussu.

Definition 1.2. An algorithm for the solution of the problem of the existence of a holomorphic first integral is a function

$$\mathcal{G} : N \mapsto 2^{J^N(2)} \times 2^{J^N(2)} \times 2^{J^N(2)}$$

such that $\mathcal{G}(N) = (M_+^N, M_-^N, M^N)$ is a decomposition of J^N (2) in piecewise disjoint sets. We say that the problem we are dealing with is algebraically solvable in the class W with respect to $\mathcal{G}$ if for every positive integer N the properties (1) and (2) in Definition 1.1 hold true for M_+^N, M_-^N, M^N.

Our main result is

THEOREM 1.1. *The problem of the existence of a holomorphic first integral of* (1.1) *is algebraically not solvable with respect to the Mattei and Moussu algorithm* (*cf.* *Section* 2) *in a class* W *of germs of real analytic* 1-*forms at* $O \in \mathbf{R}^2$, *all having real dynamics of center type, and with first non-vanishing jet of the form* $(x^2 + y^2)\{x\,dx + y\,dy\}$.

Remark 1.3. We explicitly observe that this result is different from the one about the algebraic solvability of the problem we consider. Actually, this depends on the choice of the function $\mathcal{G}$ in Definition 1.2. The definition of $\mathcal{G}$ relative to the Topological Criterion by Mattei and Moussu and the motivation for this choice are given in Section 2.

The paper is organized as follows: in Section 2 we rapidly recall the basic tools in the study of a singularity of a differential equation in $\mathbf{C}^2$, such as the reduction of a singularity and the holonomy of a leaf of a foliation. Then we use the fundamental Topological Criterion by J. F. Mattei and R. Moussu to give the definition and the analytic characterization of the sets M_+^N, M_-^N, M^N.

In Section 3 we define the class W in the statement of Theorem 1.1, and we prove the theorem by showing that M_+^5, M_-^5, M^5 cannot satisfy the first condition in Definition 1.1.

This paper originated from a question posed by R. Conti. The unsolved[1] conjecture of the algebraic non-solvability of the problem of the existence of a holomorphic first integral is due to Yu.S. Il'yashenko: the author thanks him and R. Moussu for stimulating conversations.

2. Analytic Conditions for the Existence of Holomorphic First Integrals

A fundamental result in the study of complex foliations is the Topological Criterion by J. F. Mattei and R. Moussu.

[1] *Note Added in Proof.* The conjecture of nonalgebraic solvability of the problem of existence of a holomorphic first integral has been proved by the author in a paper to appear in the *Journal of Dynamical and Control System.*

THEOREM 2.1. (Topological Criterion [5]). *Let ω be a holomorphic 1-form, and let $O \in \mathbf{C}^2$ be an isolated singular point of ω. Let $\mathcal{F}_\omega$ be the foliation induced by ω. Then ω has a holomorphic first integral if and only if $\mathcal{F}_\omega$ is* simple, *i.e., if there exists a neighborhood U of $O \in \mathbf{C}^2$ such that*

(1) *every leaf $\mathcal{L}$ of $\mathcal{F}_\omega$ is closed in $U - \{0\}$.*

(2) *only finitely many leaves accumulate at $O \in \mathbf{C}^2$.*

The aim of this section is to obtain an analytic characterization of the set $M_-^N \cup M^N$, for any given positive integer. This goal will be obtained through a careful study of the proof of Theorem 2.1. First of all, we need to overview some basic tools, namely the blowing up of a 1-form and the holonomy of a leaf of a foliation.

Let $\omega = 0$ be a holomorphic differential equation on $U \subset \mathbf{C}^2$, $O \in U$, with an isolated singular point at the origin. Let $\mathcal{F} = (U, \{0\})$ be its associated singular foliation. The *resolution* of the singularity $O \in \mathbf{C}^2$ of $\mathcal{F}_\omega$ is a new singular foliation

$$\tilde{\mathcal{F}}_\omega = (V, \operatorname{sing} \tilde{\mathcal{F}}_\omega)$$

defined in a neighbourhood V of an analytic set D, the *divisor* of the resolution, such that

$$\operatorname{sing} \tilde{\mathcal{F}}_\omega \subset D.$$

The divisor D is a chain of n projective lines P_k, such that any two P_k's intersect trasversally at most at one point: $\{m_k\} = P_k \cap P_{k-1}$. The link between $\mathcal{F}_\omega$ and $\tilde{\mathcal{F}}_\omega$ is the holomorphic map

$$\pi : (V, D) \mapsto (\mathbf{C}^2, O) \tag{2.1}$$

defined through the composition of blowing ups (see [5] for the definitions)

$$\pi_k : (V^{(k)}, D_k) \mapsto (V^{k-1}, D_{k-1}).$$

The map

$$\pi_{|V-D} : V - D \mapsto \mathbf{C}^2 - \{0\}$$

is a diffeomorphism.

We say that $\tilde{\mathcal{F}}_\omega$ is *non-dicritic* (see [5]) if $\#\operatorname{sing} \tilde{\mathcal{F}}_\omega < \infty$: if $\tilde{\mathcal{F}}_\omega$ is non-dicritic, for each k, $k = 1, 2, \ldots, n$ we have

$$P_k = \mathcal{P}_k \cup \{p_1^{(k)}, \ldots, p_{jk}^{(k)}\},$$

where $\mathcal{P}_k$ is a leaf of $\tilde{\mathcal{F}}_\omega$, the *k-projective leaf*, and $p_1^{(k)}, \ldots, p_{j_k}^{(k)}$ are singular points of the foliation. For each P_k we define a negative integer $S(P_k)$: the *self intersection number* of P_k (see [6]). In each $V^{(k)}$ the foliation $\tilde{\mathcal{F}}_\omega$ is defined through a 1-form $\omega^{(k)}$ which can be described in two local charts. The singular points of the $\omega^{(k)}$'s turn to be *elementary*, i.e., the dual vector field of $\omega^{(k)}$ has at each singular point a linearization with at least one non-zero eigenvalue. If both the eigenvalues are non-zero, the singular point is a *saddle* and the ratio λ of the eigenvalues satisfies $\lambda \notin \mathbf{Q}_+$, if $\tilde{\mathcal{F}}_\omega$ is non-dicritic.

Moreover if $\lambda \in \mathbf{Q}_-$, the singular point is a *resonant saddle.* If one of the two eigenvalues is zero, the singular point is a *saddle-node.*

A saddle defines a local foliation with two separatrices: a *separatrix* is a leaf of a foliation such that it is an analytic set after the singular point is added to it.

The fundamental Dulac–Seidenberg Theorem guarantees that every holomorphic differential equation with an isolated singular point admits a resolution obtained after a finite number of blowing ups.

If $\mathcal{F}$ is a foliation and $\mathcal{L}$ is a leaf of $\mathcal{F}$, for a given $p \in \mathcal{L}$ and for every $\gamma \in \pi_1(\mathcal{L}, p)$ it is defined a germ of conformal map h_γ, the holonomy of $\mathcal{F}$ with respect to $\mathcal{L}$ and γ: a representant of h_γ is the first return map, or Poincaré map, defined by the lifting of the path γ through the foliation (see [5]).

The map

$$\mathcal{H} = \mathcal{H}(\mathcal{F}, \mathcal{L}; p) : \pi_1(\mathcal{L}; p) \mapsto \operatorname{Diff}(\mathbf{C}, 0),$$

where Diff ($\mathbf{C}$, 0) is the group of germs of conformal mappings, turns out to be a homomorphism of groups.

The subgroup of Diff ($\mathbf{C}$, 0) defined as

$$H(\mathcal{F}, \mathcal{L}; p) := \mathcal{H}(\mathcal{F}, \mathcal{L}; p)(\pi_1(\mathcal{L}, p))$$

is the *holonomy group* of the leaf $\mathcal{L}$ of $\mathcal{F}$.

For every projective leaf, $\mathcal{P}_k$, of the resolution of a non-dicritic foliation we define the *k-projective holonomy* as

$$H_k := H(\tilde{\mathcal{F}}_\omega, \mathcal{P}_k; p)$$

with $p \in \mathcal{P}_k$.

We end this overview on the basic tools of the local theory of holomorphic foliations in $\mathbf{C}^2$ by considering an algebraic invariant which is very important in connection with the problem of extension of local holomorphic first integrals of a foliation.

Let $f\colon (\mathbf{C}, 0) \mapsto (\mathbf{C}, 0)$ be the germ of a holomorphic function. We define the *group of invariance* of f (see [5]) as

$$H(f) := \{h \in \operatorname{Diff}(\mathbf{C}, 0) \mid f \circ h = f\}.$$

Let $\mathcal{C}$ be the class of foliations having an isolated singular point at $O \in \mathbf{C}^2$ and admitting a holomorphic first integral. We will give now an algorithmic description of the class $\mathcal{C}$ by giving a brief description of the proof of Theorem 2.1. Let $\mathcal{F}_\omega$ be a given foliation with $O \in \mathbf{C}^2$ as isolated singular point and let $\tilde{\mathcal{F}}_\omega$ be its resolution.

Step 0. *Non-dicriticness.* If $\tilde{\mathcal{F}}_\omega$ is dicritic, $\mathcal{F}_\omega \notin \mathcal{C}$. (Note that the condition that $\tilde{\mathcal{F}}_\omega$ is non-dicritic can be easily checked by algebraic computations: in fact to claim that $\tilde{\mathcal{F}}_\omega$ is dicritic means that there exists a $\omega^{(k)}$ in the resolution such that if h is the *order* of $\omega^{(k)}$, i.e., the order of its first non-vanishing jet, and

$$\omega^{(k)}(x, y) = (A_h^{(k)}(x, y) + \cdots)\, dx + (B_h^{(k)}(x, y) + \cdots)\, dy$$

then $xA_h^{(k)}(x, y) + yB_h^{(k)}(x, y) \equiv 0$.)

Step 1. *Local Obstruction.* If $p \in \operatorname{sing} \tilde{\mathcal{F}}_\omega$ and p is not a resonant saddle with periodic holonomy, then $\mathcal{F}_\omega \notin \mathcal{C}$. Otherwise we can define in a neighbourhood of any singular point of $\tilde{\mathcal{F}}_\omega$ a local holomorphic first integral (see [5, Theorem B_0, p. 480]).

Step 2. *Global Obstruction.* Let P_k be an element of the chain forming D, with $S(P_k) = -1$. Let $p_1^{(k)}, \ldots, p_{jk}^{(k)} \in \operatorname{sing} \tilde{\mathcal{F}}_\omega \cap P_k$. The *group of invariance* of $\mathcal{P}_k$ is defined as the group generated by the groups of invariance of each local first integral near the $p_i^{(k)}$'s (see previous step): we call it H^k.

Remark 2.1. It is straightforward to see that for every $k = 1, 2, \ldots, n$

$$H_k \leq H^k,$$

i.e., the k-projective holonomy group is a subgroup of H^k.

In [5] the following is proved

PROPOSITION 2.1. *The foliation $\tilde{\mathcal{F}}_\omega$ admits a holomorphic first integral defined in a neighbourhood $V^{(k)}$ of P_k if and only if H^k is abelian (equivalently: if and only if H^k is finite).*

In particular, if there exists a P_k such that $S(P_k) = -1$ and H^k is not abelian, the $\mathcal{F}_\omega \notin \mathcal{C}$. Otherwise, for every P_k with $S(P_k) = -1$ we can define a local holomorphic first integral in the respective $V^{(k)}$.

Step 3. *Recursive Step.* If the algorithm does not end at the second step we blow down $\tilde{\mathcal{F}}_\omega$ obtaining a new singular foliation having local holomorphic first interal defined in sufficiently small neighbourhoods of every singular point, and with projective leaves with the self intersection number increased by one with respect to those of $\tilde{\mathcal{F}}_\omega$. Now we rename this new singular foliation as $\tilde{\mathcal{F}}_\omega$ and we repeat Step 2.

Conclusion. The above algorithm ends after a finite number of steps leading to one of these three possible situations:

(a) the local test in Step 1 fails: $\mathcal{F}_\omega \notin \mathcal{C}$.

(b) the global test in Step 2 fails: $\mathcal{F}_\omega \notin \mathcal{C}$.

(c) neither the local nor the global test ever fail: $\mathcal{F}_\omega \in \mathcal{C}$ (this is Theorem 2.1), see [5]).

We will use this algorithmical description of the Mattei and Moussu Topological Criterion to give a characterization of the set $M_+^N \cup M^N$. We need some more definitions. Let $J^N(2)$ be the vector space of N-jets of analytic 1-forms defined in a neighbourhood of $O \in \mathbf{R}^2$.

DEFINITION 2.1. Let $\tilde{\omega} \in J^N(2)$: $\tilde{\omega}$ is *equidesingularizable* if for every analytic 1-form ω such that $J^N \omega = \tilde{\omega}$ we have

(a) ω has an isolated singular point at $O \in \mathbf{R}^2$.

(b) The resolution of the complexification of ω is obtained after the same chain of blow ups, independently of the continuation of $\tilde{\omega}$. We call this resolution the *resolution of* $\tilde{\omega}$.

Let W be a class of analytic germs of 1-forms at $O \in \mathbf{R}^2$, and let $O \in \mathbf{R}^2$ be an isolated singular point for each $\omega \in W$. Let N be a positive integer. The class W is *N-equidesingularizable* if for every $\tilde{\omega} \in W^N$, $\tilde{\omega}$ is equidesingularizable and the resolutions of all the $\tilde{\omega}$'s in W^N are obtained after the same chain of blow ups.

Remark 2.2. For instance, the class $W = \{\omega \mid J^1\omega = x\,dx + y\,dy\}$ is N-equidesingularizable for every $N \geq 1$.

DEFINITION 2.2. Let $\tilde{\omega}$ be a equidesingularizable N-jet. For each projective leaf $\mathcal{P}_k$ and for each positive integer m we define the *m-approximate* H_k as

$$J^m(H_k) := \{\tilde{h} \in J^m(1) \mid \tilde{h} = J^m h, \text{for some } h \in H_k\}.$$

Here $J^m(1)$ is the space of m-jets of germs of conformal maps.

PROPOSITION 2.2 *Let $\tilde{\omega}$ be a equidesingularizable N-jet. Then for every projective leaf it is possible to compute $J^m(H_k)$ for every $m \leq \overline{m}(\tilde{\omega}, k)$ ($\overline{m}(\tilde{\omega}, k)$ positive integer, possibly $\overline{m}(\tilde{\omega}, k) = \infty$).*

We do not prove the above proposition in full generality, i.e., we do not find an explicit expression for $\overline{m}(\tilde{\omega}, k)$: we only use the fact that such an $\overline{m}$ exists. This is easy to prove: just perform the necessary blow ups and study the approximations of the differential equations which locally define the foliation. A computation of this type is done in the case of interest for us in Section 3 (see Lemma 3.2).

DEFINITION 2.3. Let W be an $\tilde{N}$-equidesingularizable class of germs of 1-forms. A function $\mathcal{G}:\mathbf{N} \mapsto 2^{J^N(2)} \times 2^{J^N(2)} \times 2^{J^N(2)}$, $\mathcal{G}(N) = (M_+^N, M_-^N, M^N)$ is a Mattei and Moussu algorithm relative to the problem of existence of a holomorphic first integral if $N \geq \tilde{N}$

$$M_-^N = \{\tilde{\omega} \in W^N \mid \text{there exists a projective leaf } \mathcal{P}_k \text{ such that } J^{\overline{m}}(H_k) \text{ is not abelian}\}$$

(we remark that different choices of M_+^N, M^N give rise to different examples of Mattei and Moussu algorithms), and if $N < \tilde{N}$ then $M_-^N = W^N$.

The motivation for this choice is given by the following proposition and remark.

PROPOSITION 2.3. *Let $\tilde{\omega}$ be a equidesingularizable N-jet such that if $J^N \omega = \tilde{\omega}$ then ω satisfies the conditions:*

(1) *$\tilde{F}_\omega$ is non-dicritic*
(2) *all the singular points of $\tilde{F}$ are resonant saddles.*

Then for each $P_k \subset D$, with $S(P_k) = -1$ and such that for all the singular points in P_k there exists a local first integral, we have that

$$H_k = H^k.$$

Remark 2.3. In Definition 2.3 the negative set M_-^N is defined only through the condition on the projective holonomies, and no mention on the groups of invariance of the resolution is given. The justification of this fact is in the above proposition (if $S(P_k) = -1$ the group of invariance, provided it can be defined, coincides with the projective holonomy group) and, generally, in the fact that it is not clear how to compute any approximation $J^m(H^k)$ of H^k from the knowledge of an approximation $J^l h$ of $h \in H^k \backslash H_k$. This fact raises two problems:

—The Mattei and Moussu procedure gives us no possibility to decide if H^k is actually defined (i.e., the existence of local first integrals near the singularities of P_k, it seems to be a transcendental condition, i.e., a condition involving the knowledge of the complete Taylor expansion of the original vector field).

—The computation of $J^l h$ implies (cf. with the definition of group of invariance) the approximate solution of a functional equation of type $f \circ h = f$ where f is a rearrangement of the Taylor expansion of a first integral defined in a neighbourhood of P_k with respect to one variable. It is easy to see that if $f(x) = b_1(t)x^r + b_2(t)x^{r+1} + \cdots$ any $b_i(t)$ cannot be computed from a jet of order N of the original vector field. Then the approximate solution of the equation $f \circ h = f$ seems to be difficult to obtain.

Remark 2.4. The proposition actually says that the existence of a holomorphic first integral in a neighbourhood of a projective leaf carrying only elementary singularities is a matter of the holonomy group only.

Proof (*of Proposition* 2.3). To prove this statement it is sufficient to prove that for each singular point (resonant saddle) of P_k, $S(P_k) = -1$, the local group of holonomy and the local group of invariance coincide (see [5] for precise definitions). Let $m \in P_k$ be a resonant saddle: we can choose local coordinates (x, y) such that $x(m) = y(m) = 0$ and $\{x = 0\}$ is the local equation defining P_k. In these coordinates we have that

$$\omega^{(k)}(x, y) = g(x, y)(py\,dx + qx\,dy + \cdots), \qquad g(0) \neq 0, g(x, y) \text{ a unit,}$$

where $p, q \in \mathbf{N}$, $(p, q) = 1$. It is easy to prove (see [5, p. 499]) that a local first integral of $\omega^{(k)}$ is of the form

$$f(x, y) = x^p y^q F(x, y), \qquad F(0) \neq 0.$$

Let us define for a sufficiently small x_0, $x_0 \neq 0$

$$f^0(y) = f(x_0, y).$$

Then a straightforward computation (see [5, p. 476]) permits us to compute the order of the local group of invariance at m

$$\#H(f^0(y)) = q.$$

On the other hand, the local group of holonomy at m with respect to $\mathcal{P}_k$ is obtained as the monodromy of the differential equation $\omega^{(k)} = 0$, or equivalently as the monodromy of

$$\frac{dy}{dx} = -\frac{p}{q}\frac{y}{x} + \cdots.$$

Then $H_k(m)$ is generated by the map germ

$$h(x) = e^{-2\pi i p/q}x + \cdots.$$

Hence, as $(p, q) = 1$, order $(h) = q$ and then (see Remark 2.1) $H_k = H^k$. ∎

We end this section with a simple lemma giving a justification of the choice of the class of germs W considered in Section 3. This lemma has been suggested by Yu. S. Il'yashenko.

LEMMA 2.1. *Let* (1.2) *satisfy*

(i) *if r is the order of ω at $O \in \mathbf{C}^2$ we suppose that:*
$r > 1$,
r is odd

(ii) $xA_r(x, y) + yB_r(x, y) = \lambda(x^2 + y^2)^{(r+1)/2}$, $\lambda \in \mathbf{C} - \{0\}$

(iii) $B_r(1, i) \neq 0$.

Then (1.2) *does not admit local holomorphic first integrals at $O \in \mathbf{C}^2$.*

Proof. We can suppose $\lambda = 1$. After one blow up we obtain

$$\begin{aligned} \dot{x} &= xB_r(1, t) + x^2\{\cdots\} \\ \dot{t} &= (1 + t^2)^{(r+1)/2} + x\{\cdots\}, \end{aligned}$$

where $x = x$, $y = xt$. Linearizing at the singular point $(0, i)$ we obtain

$$\left.\frac{\partial(\dot{x}, \dot{t})}{\partial(x, t)}\right|_{(0,i)} = \begin{pmatrix} B_r(1, i) & 0 \\ * & 0 \end{pmatrix}.$$

From (iii), $(0, i)$ is a saddle-node, hence (1.2) cannot admit any holomorphic first integral. ∎

Remark 2.5. There are examples of differential equations satisfying (i)–(iii) in Lemma 2.1 and having real dynamic of center type, for instance

$$\begin{aligned} \dot{x} &= y^3 \\ \dot{y} &= x(x^2 + 2y^2). \end{aligned}$$

These differential equations gives counterexamples, different from those of the type given by Moussu in [4], of the non-existence of holomorphic first integral for differential equations having a center.

3. Algebraic Non-solvability

In this section we will prove Theorem 1.1 by giving an example of a class W of germs of analytic 1-forms with real dynamics of center type at $O \in \mathbf{R}^2$, for which $M^5_+ \cup M^5$ is not semialgebraic. We will give the definition of W step by step, justifying our choices by the arguments of the previous section.

Choice of the 3-jet

We denote a 1-form $\omega(x, y)$ as

$$\omega(x, y) = \sum_{n \geq l} \omega_n(x, y),$$

where

$$\omega_n(x, y) = A_n(x, y)\, dx + B_n(x, y)\, dy$$

and A_n, B_n are homogeneous polynomials of degree n; l is the order of ω at $O \in \mathbf{R}^2$: as O is a singular point, $l \geq 1$. From Remark 1.3 and Lemma 2.1 it follows that the 1-forms in W must have order $l \geq 3$, and that a possible choice of the 3-jet is

$$\omega_3(x, y) = (x^2 + y^2)\,(x\, dx + y\, dy).$$

The First Blow Up

Let ω be a 1-form , with $J^3\omega = (x^2 + y^2)(x\, dx + y\, dy)$. We suppose that $O \in \mathbf{C}^2$ is an isolated singular point of ω (see Remark 3.2). Blowing up ω at 0 we obtain

$$\begin{aligned} \omega^{(1)}(x, t) = {} & (1 + t^2)((1 + t^2)\, dx + tx\, dt) \\ & + x \sum_{k \geq 4} x^{k-4}\,(P_k(1, t)\, dx + xB_k(1, t)\, dt), \end{aligned} \tag{3.1}$$

where

$$P_k(x, y) = xA_k(x, y) + yB_k(x, y). \tag{3.2}$$

The singular points of $\omega^{(1)}$ on P_1 are $(0, \pm i)$. Linearizing the dual vector field of $\omega^{(1)}$, $\omega_2^{(1)}(\partial/\partial x) - \omega_1^{(1)}(\partial/\partial y)$, at the points $(0, \pm i)$, we obtain

$$\frac{\partial(\omega_2^{(1)} - \omega_1^{(1)})}{\partial(x, t)}(0, i) = \begin{pmatrix} 0 & 0 \\ -P_4(1, i) & 0 \end{pmatrix}$$

$$\frac{\partial(\omega_2^{(1)} - \omega_1^{(1)}}{\partial(x, t)}(0, i) = \begin{pmatrix} 0 & 0 \\ -P_4(1, -i) & 0 \end{pmatrix}.$$

Then $(0, \pm i)$ are non-elementary singular points, and we blow them up again. We name $\mathbf{CP}^{(1)}$ the divisor after this blow up.

The Second Blow Up

First, let us blow up $\omega^{(1)}$ at $(0, i)$. We translate $(0, i)$ in the origin of the new coordinate system $x = x$, $s = t - i$, obtaining

$$\begin{aligned} \omega^{(1)}(x, s) = {} & \left[s^2(s + 2i)^2 + x \sum_{k \geq 4} x^{k-4} P_k(s) \right] dx \\ & + x \left[s(s + 2i)(s + i) + x \sum_{k \geq 4} x^{k-4} B_k(s) \right] ds, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} P_k(s) &= P_k(1, s + i) \\ B_k(s) &= B_k(1, s + i). \end{aligned} \tag{3.4}$$

We will suppose that

$$P_4(s) = sP_4^*(s), \qquad P_4^*(0) \neq 0 \tag{3.5}$$

(see Remark 3.1 for a justification of this choice.)

Blowing up the singularity at $O \in \mathbf{C}^2$ of (3.3) and using (3.5) we obtain, in the two local charts of coordinates,

$$x = x,$$

$$s = vx$$

and

$$x = sz,$$
$$s = s$$

$$\omega^{(2)}(x, v) = \left[v^2(xv + 2i)^2 + v^2(xv + 2i)(xv + i) \right.$$
$$\left. + vP_4^*(xv) + \sum_{k\geq 5} x^{k-5} P_k(xv) + v \sum_{k\geq 4} x^{k-4} B_k(xv) \right] dx$$
$$+ x \left[v(xv + 2i)(xv + i) + \sum_{k\geq 4} x^{k-4} B_k(xv) \right] dv \tag{3.6}$$

$$\omega^{(2)}(z, s) = s\left[(s + 2i)^2 + zP_4^*(s) + z \sum_{k\geq 5} z^{k-4}s^{k-5} P_k(s) \right] dz$$
$$+ z \left[(s + 2i)(2s + 3i) + zP_4^*(s) + z \sum_{k\geq 5} z^{k-4}s^{k-5} P_k(s) \right.$$
$$\left. + z \sum_{k\geq 4} (zs)^{k-4} B_k(s) \right] ds. \tag{3.7}$$

We put

$$\begin{aligned} P_k(s) &= \sum_{j=0}^{k+1} p_j^{(k)} s^j \\ B_k(s) &= \sum_{j=0}^{k} b_j^{(k)} s^j. \end{aligned} \tag{3.8}$$

Then (3.5) becomes

$$p_0^{(4)} = 0, \qquad p_1^{(4)} \neq 0. \tag{3.9}$$

The singular point $(0, i)$ has been blown up in the projective line $\mathbf{CP}^{(2)}$, and the dynamics of $\omega^{(2)}$ along $\mathbf{CP}^{(2)}$ are described by

$$\begin{aligned} \omega^{(2)}(z, 0) &= z[-6 + z[p_1^{(4)} + b_0^{(4)}] + z^2 p_0^{(5)}]\, ds \\ \omega^{(2)}(0, v) &= [-6v^2 + v(p_1^{(4)} + b_0^{(4)}) + p_0^{(5)}]\, dx. \end{aligned} \tag{3.10}$$

We perform now the same blow up at the singular point $(0, -i)$ of $\omega^{(1)}$. Let $x = x$ and $s = t + i$: then $\omega^{(1)}(x, s)$ is singular at $0 \in \mathbf{C}^2$. We define

$$\begin{aligned} \tilde{P}_k(s) &= P_k(1, s-i) \\ \tilde{B}_k(s) &= B_k(1, s-i), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \tilde{P}_k(s) &= \sum_{j=0}^{k+1} \tilde{p}_j^{(k)} s^j \\ \tilde{B}_k(s) &= \sum_{j=0}^{k} \tilde{b}_j^{(k)} s^j. \end{aligned} \tag{3.12}$$

We suppose again that $\tilde{P}_4(s) = s\tilde{P}_4^*(0) \neq 0$, or that

$$\tilde{p}_0^{(4)} = 0, \qquad \tilde{p}_1^{(4)} \neq 0. \tag{3.13}$$

Remark 3.1. If $p_0^{(4)} \neq 0$ and $\tilde{p}_0^{(4)} \neq 0$ it is easy to see that the resolution of the singular points (that we suppose to be isolated) of $\omega^{(1)}$ need at least another step of blowing up to get the resolution of ω.

The singular point $(0, -i)$ of $\omega^{(1)}\,(x, t)$ has now been blown up obtaining a 1-form $\omega^{(3)}$ in a neighbourhood of $\mathbf{CP}^{(3)}$. The analytic expansions of $\omega^{(3)}$ in local coordinates are

$$\begin{aligned} \omega^{(3)}(x, v) = \Bigg[& v^2(xv - 2i)^2 + v\tilde{P}_4^*(xv) + \sum_{k\geq 5} x^{k-5}\,\tilde{P}_k(xv) \\ & + v^2(xv - 2i)(xv - i) + v \sum_{k\geq 4} x^{k-4}\,\tilde{B}_k(xv) \Bigg] dx \\ & + x \left[v(xv - 2i)(xv - i) + \sum_{k\geq 4} x^{k-4}\,\tilde{B}_k(xv) \right] dv \end{aligned} \tag{3.14}$$

$$\begin{aligned} \omega^{(3)}(z, s) = s\Bigg[& (s - 2i)^2 + z\tilde{P}_4^*(s) + z \sum_{k\geq 5} z^{k-4} s^{k-5}\,\tilde{P}_k(s) \Bigg] dz \\ & + z \Bigg[(s - 2i)(2s - 3i) + z\tilde{P}_4^*(s) + z \sum_{k\geq 5} z^{k-4} s^{k-5}\,\tilde{P}_k(s) \\ & + z \sum_{k\geq 4} (zs)^{k-4}\,\tilde{B}_k(s) \Bigg] ds. \end{aligned} \tag{3.15}$$

The dynamics of the differential equation $\omega^{(3)} = 0$ along $\mathbf{CP}^{(3)}$ are given by

$$\begin{aligned} \omega^{(3)}(0, v) &= [-6v^2 + v(\tilde{p}_1^{(4)} + \tilde{b}_0^{(4)}) + \tilde{p}_0^{(5)}]\, dx \\ \omega^{(3)}(z, 0) &= z\,[-6 + z(\tilde{p}_1^{(4)} + \tilde{b}_0^{(4)}) + z^2\tilde{p}_0^{(5)}]\, ds. \end{aligned} \tag{3.16}$$

From (3.10) and (3.16) it follows that the singular points on $D = \mathbf{CP}^{(1)} \cup \mathbf{CP}^{(2)} \cup \mathbf{CP}^{(3)}$ are given by

$$z = s = 0; \qquad x = 0, \qquad v_{1,2} = \frac{-(p_1^{(4)} + b_0^{(4)}) \pm \sqrt{(p_1^{(4)} + b_0^{(4)})^2 + 24p_0^{(5)}}}{12} \tag{3.17}$$

in $\mathbf{CP}^{(2)}$, and

$$z = s = 0; \qquad x = 0, \qquad \tilde{v}_{1,2} = \frac{-(\tilde{p}_1^{(4)} + \tilde{b}_0^{(4)}) + \sqrt{(\tilde{p}_1^{(4)} + \tilde{b}_0^{(4)})^2 + 24\tilde{p}_0^{(5)}}}{12} \tag{3.18}$$

in $\mathbf{CP}^{(3)}$. Linearizing the differential equations $\omega^{(2)} = 0$ and $\omega^{(3)} = 0$ near these six singular points we obtain

$$\left.\frac{\partial(\dot{x}, \dot{y})}{\partial(x, y)}\right|_{(0,v_i)\in\mathbf{CP}^{(2)}} = \begin{pmatrix} -2v_i + b_0^{(4)} & 0 \\ * & 12v_i - (p_1^{(4)} + b_0^{(4)}) \end{pmatrix} \tag{3.19}$$

$$\left.\frac{\partial(\dot{x}, \dot{y})}{\partial(x, v)}\right|_{(0,\tilde{v}_i)\in\mathbf{CP}^{(3)}} = \begin{pmatrix} -2\tilde{v}_i + \tilde{b}_0^{(4)} & 0 \\ * & 12\tilde{v}_i - (\tilde{p}_1^{(4)} + \tilde{b}_0^{(4)}) \end{pmatrix} \tag{3.20}$$

$i = 1, 2$ and

$$\left.\frac{\partial(\dot{z}, \dot{s})}{\partial(z, s)}\right|_{(0,0)\in\mathbf{CP}^{(2)}} = \left.\frac{\partial(\dot{z}, \dot{s})}{\partial(z, s)}\right|_{(0,0)\in\mathbf{CP}^{(3)}} = \begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}. \tag{3.21}$$

The Definition of W

Let us define

$$\omega_k(x, y) = \left(\sum_{j=0}^{k} a_{k-j,j}x^{k-j}y^j\right) dx + \left(\sum_{j=0}^{k} b_{k-j,j}x^{k-j}y^j\right) dy. \tag{3.22}$$

Let us recall (see [7, p. 199]) that if $g(x, y) = (x, -y)$, a 1-form ω, with an isolated singular monodromic point at $0 \in \mathbf{R}^2$, has real dynamic of center type if

$$g^*\omega = \omega. \tag{3.23}$$

DEFINITION 3.1. Let

$$W = \{\omega \text{ is a germ at } O \in \mathbf{R}^2 \text{ of an analytic 1-form, satisfying (1)–(4)}\},$$

where

(1) $0 \in \mathbf{C}^2$ is an isolated singular point of the complexification of ω
(2) $g^*\omega = \omega$
(3) $\omega_3(x, y) = (x^2 + y^2)(x\,dx + y\,dy)$
(4) referring to (3.22) the following semialgebraic conditions hold

$$\begin{aligned}
b_{31} &= b_{13}\\
b_{31} + a_{22} &= a_{40}\\
a_{04} + b_{13} &= 0\\
a_{40} &\neq 0\\
p_0^{(5)} &< 0\\
\pm p_1^{(4)} \pm \sqrt{(p_1^{(4)})^2 + 24p_0^{(5)}} &\neq 0\\
\pm 2p_1^{(4)} \pm \sqrt{(p_1^{(4)})^2 + 24p_0^{(5)}} &\neq 0\\
\frac{12v_i - p_1^{(4)}}{2v_i} &< 0\\
\frac{12\tilde{v}_i - \tilde{p}_1^{(4)}}{2\tilde{v}_i} &\leq 0.
\end{aligned} \tag{3.24}$$

Remark 3.2. The condition (1) is compatible with (2) and (3): for instance a (germ of a) 1-form

$$\omega(x, y) = (x^2 + y^2)(x\,dx + y\,dy) + \omega_k(x, y) + \cdots + \omega_n(x, y) + \cdots$$

such that each $\omega_k(x, y)$ satisfies $g^*\omega_k = \omega_k$ and in particular $\omega_n(x, y) = y^n\,dx$, n even, $n > 5$, verifies (1)–(3) in the definition of W. Let us explicitly note that if ω satisfies (1)–(3) it has a center at $O \in \mathbf{R}^2$. Moreover the condition (2) implies that $g^*\omega_4 = \omega_4$ and $g^*\omega_5 = \omega_5$ and then

$$a_{31} = a_{13} = b_{40} = b_{22} = b_{04} = a_{41} = a_{23} = a_{05} = b_{05} = b_{32} = b_{14} = 0. \quad (3.25)$$

Then a straightforward computation leads to

$$p_0^{(5)} = \tilde{p}_0^{(5)} = (a_{50} - a_{32} + a_{14}) + b_{23} - b_{41} - b_{05}) \quad (3.26)$$

and so (3.24) makes sense. Moreover, it is worth noting that the condition $b_{31} = b_{13}$ in (3.24) implies

$$b_0^{(4)} = -\tilde{b}_0^{(4)} = 0 \quad (3.27)$$

while the conditions $b_{31} + a_{22} = a_{40}$, $a_{04} + b_{13} = 0$, and $a_{40} \neq 0$ imply that

$$p_1^{(4)} = -\tilde{p}_1^{(4)} = 2ia_{40} \neq 0. \quad (3.28)$$

Finally to conclude that $W \neq \varnothing$ we have to show that the two last inequalities in (3.24) make sense and define a non-empty set in $J^5(2)$: this is a consequence of (3.28) which implies that $(12v_i - p_1^{(4)})/2v_i$, $(12\tilde{v}_i - \tilde{p}_1^{(4)})/2\tilde{v}_i \in \mathbf{R}$, and of

$$\lim_{-p_0^{(5)} \to \infty} \frac{12v_i - p_1^{(4)}}{2v_i} = \lim_{-p_0^{(5)} \to \infty} \frac{12\tilde{v}_i - \tilde{p}_1^{(4)}}{2\tilde{v}_i} = -\frac{1}{6}.$$

LEMMA 3.1. (a) *The projection W^5 of W on J^5 (2) is a semialgebraic subset of J^5 (2).*

(b) *The class W is N-equidesingularizable* (*see Definition* 2.1) *for each $N \geq 5$. Moreover every singular point in the resolution of $\omega \in W$ is a saddle with eigenvalues in the Siegel domain* (*i.e., with negative ratio*).

Proof. The statement (a) is a straightforward consequence of (3.25) and (3.24).

To prove the statement (b) we observe that from (3.17)–(3.21), from (3.26)–(3.28), and from (3.24), it follows that after the blowing ups of the singular points $(0, \pm i)$ of $\omega^{(1)}$ we have obtained a foliation in a neighbourhood of $D = \mathbf{CP}^{(1)} \cup \mathbf{CP}^{(2)} \cup \mathbf{CP}^{(3)}$, with two corners which are saddles (see (3.21)) and with four other singular points: these singular points are saddles, with eigenvalues having negative ratio, as follows from the last conditions in (3.24). ∎

Characterization of $M^5_+ \subset M^5$ in W

From Lemma 3.1 it follows that every $\tilde{\omega} \in W^5$ has the same resolution: in other words if $J^5(\omega) = \tilde{\omega}$ then $(\tilde{\mathcal{F}}_\omega, \operatorname{sing} \tilde{\mathcal{F}}_\omega)$ is defined through $(\omega^{(1)}, V^{(1)})$, $(\omega^{(2)}, V^{(2)})$, and $(\omega^{(3)}, V^{(3)}, V^{(3)})$, where the $V^{(i)}$'s are neighbourhoods of the $\mathbf{CP}^{(i)}$'s, $i = 1, 2, 3$, and $\{m_{0j}, m_{1j}, m_{2j} \mid j = 1, 2\} = \operatorname{sing} \tilde{\mathcal{F}}_\omega$, where $\{m_{01}\} = \mathbf{CP}^{(1)} \cap \mathbf{CP}^{(2)}$ and $\{m_{02}\} = \mathbf{CP}^{(1)} \cap \mathbf{CP}^{(3)}$. Moreover, all these singular points are saddles. Referring to Definition 2.2 and Proposition 2.2 we have (recall that H_i is the projective holonomy group of $\mathbf{CP}^{(i)}$).

LEMMA 3.2. *Let $\tilde{\omega} \in W^5$.*

(a) $\overline{m}_2 = \overline{m}(\tilde{\omega}, 2) = \overline{m}_3 = \overline{m}(\tilde{\omega}, 3) = 1$, *i.e., the best approximation of H_2 and H_3 we can compute from the knowledge of $\tilde{\omega} \in W^5$ are $H_{2,1}$ and $H_{3,1}$.*

(b) *H_1 is always periodic, and actually*

$$\mathrm{H}_1 = \{\sigma, 1_{\mathbf{C}}\},$$

where $\sigma \neq 1_{\mathbf{C}}$, $\sigma^2 = 1_{\mathbf{C}}$.

Proof. Let ω be such that $J^5\,\omega = \tilde{\omega} \in W$.

To prove the statement (a) we need some computations. Let us consider the differential equation

$$\omega^{(2)}(z, s) = \omega_1^{(2)}(z, s)\, dx + \omega_2^{(2)}(z, s)\, ds$$

and let $\gamma : [0, 1] \mapsto \mathcal{P}^{(2)}$ be a closed path based at $p_0 \in \mathcal{P}^{(2)} = \mathbf{CP}^{(2)} - \operatorname{sing} \tilde{\mathcal{F}}_\omega$. To compute the holonomy of $\omega^{(2)}$ with respect to γ we consider the differential equation

$$\frac{ds}{dz} = -\frac{\omega_1^{(2)}(z, s)}{\omega_2^{(2)}(z, s)}$$

in a neighbourhood V of γ in $\mathbf{C}^2$. If

$$\omega_1^{(2)}(z, s) = \alpha_1(z)s + \alpha_2(z)s^2 + \cdots$$
$$\omega_2^{(2)}(z, s) = \beta_0(z) + \beta_1(z)s + \cdots.$$

then

$$\frac{ds}{dz} = -\left(\frac{\alpha_1}{\beta_0}(z)s + \left(\frac{\alpha_2}{\beta_0}(z) - \frac{\alpha_1\beta_1}{\beta_0^2}(z)\right)s^2 + \cdots\right). \tag{3.29}$$

For sufficiently small initial conditions $s(0, s_0) = s_0$ the solutions of (3.29) can be expanded as

$$s(z, s_0) = u_1(z)s_0 + u_2(z)s_0^2 + \cdots, \tag{3.30}$$

where the functions $u_i(z)$ are solutions of the recurrent system

$$\begin{aligned} \dot{u}_1 &= -\frac{\alpha_1}{\beta_0}(z)\, u_1 \\ \dot{u}_2 &= -\frac{\alpha_1}{\beta_0} u_2 - \left(\frac{\alpha_2}{\beta_0}(z) - \frac{\alpha_1\beta_1}{\beta_0^2}(z)\right) u_1^2 \end{aligned} \tag{3.31}$$

with initial conditions $u_1(0) = 1$, $u_j(0) = 0$, $j = 2, 3, \ldots$.

The holonomy h_γ is (the germ of) the diffeomorphism

$$h_\gamma(s_0) = s(\gamma(1), s_0)$$

and from (3.29)–(3.31) it follows that

$$h_\gamma(s_0) = h_\gamma'(0)s_0 + h_\gamma''(0)s_0^2 + \cdots,$$

where

$$h_\gamma'(0) = \exp\left(-\int_\gamma (\alpha_1/\beta_0)(z)\, dz\right) \tag{3.32}$$

$$\begin{aligned} h_\gamma''(0) = \exp\left(-\int_\gamma (\alpha_1/\beta_0)(z)\, dz\right) \int_\gamma \left(\frac{\alpha_2}{\beta_0}(z) - \frac{\alpha_1\beta_1}{\beta_0^2}(z)\right) \\ \times \exp\left(\int_0^z (\alpha_1/\beta_0)(\zeta)\, d\zeta\right) dz. \end{aligned} \tag{3.33}$$

An easy computation leads to (see (3.8), (3.12))

$$\begin{aligned} \beta_0(z) &= z[-6 + zp_1^{(4)} + z^2p_0^{(5)}] \\ \beta_1(z) &= z[7i + z(p_2^{(4)} + b_1^{(4)}) + z^2[p_1^{(5)} + b_0^{(5)}] + z^3p_0^{(6)}] \\ \alpha_1(z) &= -4 + zp_1^{(4)} + z^2p_0^{(5)} \\ \alpha_2(z) &= 4i + zp_2^{(4)} + z^2p_1^{(5)} + z^3p_0^{(6)}. \end{aligned} \tag{3.34}$$

Then from (3.32)–(3.34) it follows that $h'_\gamma(0)$ is completely determined by $\tilde{\omega} = J^5\omega$; on the contrary, it is not possible to compute $h''_\gamma(0)$ from $\tilde{\omega} = J^5\omega$.

With the same computations (we have only to substitute $\tilde{p}_j^{(k)}$, $\tilde{b}_j^{(k)}$ to $p_j^{(k)}$, $b_j^{(k)}$ in (3.34)) we obtain an analogous result for $\omega^{(3)}$: the proof of statement (a) is concluded.

To prove statement (b) we observe that the group $\pi_1(\mathcal{P}^{(1)}; p_0)$, $p_0 \in \mathcal{P}^{(1)}$, $\mathcal{P}^{(1)} = \mathbf{CP}^{(1)} - \{m_{0,1}, m_{0,2}\}$, is generated by only one loop $\gamma :< \gamma >= \pi_1(\mathcal{P}^{(1)}; p_0)$.

Then $H_1 =< h_\gamma >$. From the hypothesis that $O \in \mathbf{R}^2$ is a center for $\omega \in W$, it follows (see also [4]) that h_γ is a periodic involution, i.e., $h_\gamma = \sigma$, $\sigma \neq 1_C$, $\sigma^2 = 1_C$. ▮

To give a characterization of $M_+^5 \cup M^5$ we need to define an important invariant of a foliation, the Camacho–Sad Index (see [6]): we define it with reference to a projective leaf $\mathcal{P}_k$ of a foliation $\mathcal{F}$. Let $p \in P_k$ be a singular point and let (ξ, η) be local coordinates such that $(\xi, \eta)(p) = O \in \mathbf{C}^2$ and $\eta_{|P_k} = 0$. Then $\mathcal{F}$ is locally defined by

$$\frac{d\eta}{d\xi} = \eta f(\xi, \eta),$$

where f is a meromorphic function, and $1/f$ does not vanish along the line $\eta = 0$. The *Camacho–Sad Index* of $\mathcal{F}$ relative to $\mathcal{P}_k$ and p is defined as

$$i_p(\mathcal{P}_k; \mathcal{F}) = \mathrm{res}_0(f_{|\eta=0}).$$

The only two properties of the Camacho–Sad Index that we will need are contained in the following

LEMMA 3.3 [6, 8]. (a) *If p is a saddle then $i_p(\mathcal{P}_k; \tilde{\mathcal{F}}) = \lambda$ where λ is the* characteristic number *of the singular point, i.e., λ is the ratio of the eigenvalues at p, with the eigenvalue corresponding to the eigenspace tangent to $\mathcal{P}_k$ at the denominator.*

(b) *If p is blown up and the resulting foliation $\tilde{\mathcal{F}}^{(1)}$ is defined in a neighbourhood of the projective line $\mathbf{CP}_p$ and* sing $\tilde{\mathcal{F}}^{(1)} \cap \mathbf{CP}_p = \{q_1, ..., q_r\}$, *then*

$$\sum_{j=1}^{r} i_{q_j}(\tilde{\mathcal{F}}^{(1)}, \mathbf{CP}_p - \{q_1, ..., q_r\}) = -1.$$

Let $\lambda_{ij} = \lambda(m_{ij})$ be the characteristic number of the singular point m_{ij}, $i = 0, 1, 2$, $j = 1, 2, 3$. From (3.17)–(3.21) and from Definition 3.1 (see also (3.26)–(3.28)) we have that

$$\lambda_{01} = \lambda_{02} = -\frac{2}{3} \tag{3.35}$$

$$\frac{1}{\lambda_{11}} = -6\left(1 - \frac{p_1^{(4)}}{-p_1^{(4)} + \sqrt{(p_1^{(4)})^2 + 24p_0^{(5)}}}\right)$$

$$\frac{1}{\lambda_{21}} = -6\left(1 + \frac{p_1^{(4)}}{p_1^{(4)} + \sqrt{(p_1^{(4)})^2 + 24p_0^{(5)}}}\right). \tag{3.36}$$

Remark 3.3. From (3.24) and (3.28) it follows that $\lambda_{ij} \in \mathbf{R}$, $i = 0, 1, 2$, $j = 1, 2, 3$.

Let $\lambda_{ij}(a_{40}, p_0^{(5)})$ be functions defined in W^5. We observe that, for every fixed $a_{40} \neq 0$, from (3.36) it follows that

$$\lim_{-p_0^{(5)} \to \infty} \frac{1}{\lambda_{11}} = \lim_{-p_0^{(5)} \to \infty} \frac{1}{\lambda_{21}} = -6. \tag{3.37}$$

We have the following characterization of $M^5_+ \cup M^5$:

LEMMA 3.4. (a) $M^5_+ \cup M^5 = \bigcup_{\lambda_{11}, \lambda_{21} \in \mathbf{Q} \cap]-1/3, 0[} M(\lambda_{11}, \lambda_{21})$, *where*

$$M(\lambda_{11}, \lambda_{21}) = \{\hat{\omega} = \hat{\omega}(a_{40}, p_0^{(5)}) \mid \lambda_{11}(a_{40}, p_0^{(5)}) = \lambda_{11}, \lambda_{21}(a_{40}, p_0^{(5)}) = \lambda_{21}\}$$

(b) $M^5_+ \cup M^5 \neq \varnothing$

(c) *each nonempty* $M(\lambda_{11}, \lambda_{21})$ *is a connected component of* $M^5_+ \cup M^5$.

Proof. From Lemma 3.1, W is 5-equidesingularizable: then from Definition 2.3 and Lemma 3.2 it follows that (as every $\hat{\omega} \in W^5$ defines a nondicritic resolution) if $\hat{\omega} \in W^5$

$\hat{\omega} \in M^5_+ \cup M^5$ if and only if all the singular points in the resolution of $\hat{\omega}$ are resonant saddles.

In fact, H_1 is surely periodic, and H_2 and H_3 are 1-approximately finite if and only if all the singular points of the resolution are resonant saddles.[1] Hence, the characteristic numbers of the singular points $\tilde{\mathcal{F}}_{\hat{\omega}}$ must satisfy $\lambda_{ij}(\hat{\omega}) \in \mathbf{Q}_-$, $i = 0, 1, 2$, $j = 1, 2, 3$.

From Lemma 3.3 it follows that

$$\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = -\frac{1}{3}$$

[1] In fact, two linear conformal maps $h(x) = ax$, $\varphi(x) = bx$ always commute: $[h, \varphi](x) = x$.

and then the conditions $\lambda_{ij}(\hat{\omega}) \in \mathbf{Q}_-$ are equivalent to the condition

$$\lambda_{11}, \lambda_{21} \in \mathbf{Q} \cap]-\frac{1}{3}, 0[\tag{3.38}$$

and statement (a) follows.

Statement (b) is a straightforward consequence of (3.37), and of the observation that if $a_{40} \in \mathbf{Q}$ then $(1/i)\sqrt{(2ia_{40})^2 + 24p_0^{(5)}} \in \mathbf{Q}$ and λ_{11}, $\lambda_{21} \in \mathbf{W}$ (see below for a possible choice of $p_0^{(5)}$).

Statement (c) is an easy consequence of the continuity of the functions $\lambda_{11}(a_{40}, p_0^{(5)})$ and $\lambda_{21}(a_{40}, p_0^{(5)})$. ∎

Proof of the Main Theorem. We can prove now Theorem 1.1: it will be sufficient to prove that $M_+^5 \cup M_-^5$ has infinitely many connected components. In fact, from Lemma 3.1, W^5 is semialgebraic and then, if the problem of the existence of a holomorphic first integral were algebraically solvable, M_-^5 and M^5 have to be semialgebraic, too.

From (3.36) and (3.37) it follows that it is possible to choose, for a given rational $\bar{a}_{40} \neq 0$, a sequence of negative real numbers $\{p_0^{(5)}(k)\}_k$ such that $(1/i)\sqrt{(2i\bar{a}_{40})^2 + 24p_0^{(5)}(k)} \in \mathbf{Q}$ (for instance, for an infinite sequence of positive integers k choose $p_0^{(5)}$ such that $-2\bar{a}_{40}^2 + 24p_0^{(5)}(k) = -k^2$): therefore $\lambda_{11}(k)$, $\lambda_{21}(k) \in \mathbf{Q}$ and $\lim_{k\to+\infty} \lambda_{11}(k) = \lim_{k\to+\infty} \lambda_{21}(k) = -1/6$. To every k corresponds a connected component $M(\lambda_{11}(k), \lambda_{21}(k))$ of $M_+^5 \cup M_-^5$: hence $M_+^5 \cup M_-^5$ has infinitely many connected components, and then it cannot be the union of semialgebraic sets. ∎

Finally, we have proved that the problem of the existence of a holomorphic first integral of (1.1) is not algebraically solvable with respect to the Mattei and Moussu algorithm in a class of real analytic (germs at $O \in \mathbf{R}^2$ of) 1-forms such that the origin is an isolated singular point of center type and the five order jet of the 1-forms is of the type

$$\begin{aligned}(x^2 + y^2)(x\,dx + y\,dy) &+ [(a_{40}x^4 + (a_{40}\,b_{13})x^2y^2\,b_{13}y^4)\,dx \\ &+ b_{13}xy(x^2 + y^2)\,dy] + [(a_{50}x^5 + a_{32}x^3y^2 + a_{14}xy^4)\,dx \\ &+ (b_{50}x^5 + b_{41}x^4y + b_{23}x^2y^3)\,dy],\end{aligned}$$

where $a_{40} \neq 0$, and $a_{40}, b_{13}, a_{50}, a_{32}, a_{14}, b_{50}, b_{41}, b_{23} \in \mathbf{R}$.

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